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On a Nonlinear Recurrent Relation

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Abstract We study the limiting behavior for the solutions of a nonlinear recurrent relation which arises from the study of Navier-Stokes equations (Li and Sinai in J. Eur. Math. Soc. 10(2):267–313, 2008). Some stability theorems are also shown concerning a related class of linear recurrent relations.

Keywords Navier-Stokes · Limiting behavior · Quadratic map · Recurrent relation

1 Introduction and Main Theorems

In this paper we consider the following nonlinear recurrent relation,

$$\Lambda_p(x) = \frac{1}{p} \sum_{\substack{p_1+p_2=p \\ p_1,p_2\geq 1}} f(\gamma)\Lambda_{p_1}(x)\Lambda_{p_2}(x), \qquad \gamma = \frac{p_1}{p},\ p > 1 \tag{1.1}$$

where $\Lambda_1 = x \in \mathbb{R}$ is a parameter, and $f : [0, 1] \to \mathbb{R}$ has integral 1. This quadratic recurrent relation arises from our recent study of complex blow ups of the $3D$ Navier-Stokes system [1]. There $f(\gamma)$ takes the special form $f(\gamma) = 6\gamma^2 - 10\gamma + 4$, and we need to show that for each initial value $\Lambda_1 = x$, there exists $R(x)$ such that

$$\Lambda_p(x) = R(x)^p(1 + \delta_p),$$

and $\delta_p \to 0$ as $p \to \infty$. The main object of this paper is to prove this claim for a general class of functions f including our special function.

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It is clear that $\Lambda_p(\lambda x) = \lambda^p \Lambda_p(x)$ for any $\lambda \neq 0$. Therefore it suffices for us to show that there exists some x^* such that $\Lambda_p(x^*) = 1 + \delta_p \to 1$ as $p \to \infty$. The limiting value of $\Lambda_p(x^*)$ is 1 since the function f has integral 1 over [0, 1]. Clearly $\Lambda_p(x) = (x/x^*)^p \Lambda_p(x^*) = R(x)^p(1+\delta_p)$ if we define $R(x) = x/x^*$.

We now state our conditions on the function f.

Assumption 1.1 Let $\tilde{f}(\gamma) = f(\gamma) + f(1-\gamma)$ satisfy the following:

(1) The integral of $\tilde{f}$ over [0, 1] is 2.
(2) $\tilde{f}$ is a polynomial, i.e.

$$\tilde{f}(\gamma) = \sum_{n=0}^{N} a_n \gamma^n \tag{1.2}$$

(3) The equation in α:

$$\sum_{n=1}^{N} \frac{n a_n}{n+2} \cdot \frac{1}{n+\alpha} = 1, \tag{1.3}$$

has N distinct roots $z_1, \ldots, z_N$, in $\mathbb{C} \cap \{z : -1 < \mathrm{Re}(z) < 0\}$. Denote

$$\alpha_f = 1 - \max_k \mathrm{Re}(z_k).$$

One more technical condition, which is needed for the inductive stage of our proof, and can be justified with easy numerics, will be stated in Sect. 5 (see Remark 5.2). We now state our main theorem.

Theorem 1.2 (Main theorem) *Let Assumption 1.1 and the technical condition in Sect. 5 hold. Then there exists x^* and constant $C > 0$, such that the solution $\Lambda_p(x)$ to the recurrent relation (1.1) satisfies*

$$\left|\Lambda_p(x^*) - 1\right| \le \frac{C}{p^{\alpha_f}}, \quad \forall p > 1.$$

Consequently for any x, there exists a unique number $R = x/x^*$, such that

$$\Lambda_p(x) = R^p \left(1 + \frac{C_p}{p^{\alpha_f}}\right), \quad \forall p > 1,$$

where C_p does not depend on x and is uniformly bounded in p.

Remark 1.3 *In the special case $f(\gamma) = 6\gamma^2 - 10\gamma + 4$ [1], we have $\tilde{f}(\gamma) = 12\gamma^2 - 12\gamma + 4$, and $z_{1,2} = \frac{-1 \pm \sqrt{15}i}{2}$ are the roots of (1.3). Clearly the assumptions are satisfied with $\alpha_f = 3/2$. Our theorem says that in this case*

$$\Lambda_p(x) = R^p \left(1 + O(p^{-3/2})\right).$$

More examples can be easily constructed.

The rest of this paper is organized as follows. In Sect. 2 we derive the linear system and give the main arguments. Section 3 includes the technical estimates needed to derive the linear system. Section 4 contains some theorems on the behavior of the solutions to the linearized system. Section 5 is devoted to the estimates of the nonlinear term and the proof of the main theorem. Some elementary estimates with explicit control of constants are deferred to the Appendix.

2 Preliminary Analysis and Linearization

Since the limiting value of $\Lambda_p(x^*)$ is 1, we linearize (1.1) around 1 and this gives:

$$\begin{aligned}\Lambda_p(x) = {} & \frac{1}{p}\sum_{p_1=1}^{p-1} f(\gamma) + \frac{1}{p}\sum_{p_1=1}^{p-1} \tilde{f}(\gamma)(\Lambda_{p_1}(x)-1) \\ & + \frac{1}{p}\sum_{p_1+p_2=p} f(\gamma)(\Lambda_{p_1}(x)-1)(\Lambda_{p_2}(x)-1),\end{aligned}$$

where $\tilde{f}(\gamma) = f(\gamma) + f(1-\gamma)$. Our main goal is to show that there exists x^* such that $\Lambda_p(x^*) \to 1$. This motivates us to define a_p such that $\Lambda_p(a_p) = 1$. When p tends to infinity a_p is a good approximation of x^*. Now let us write

$$\frac{a_p}{a_{p-1}} = 1 + \frac{\xi_p}{p^3}.$$

The scaling p^{-3} here is not intuitively obvious and in fact is not optimal since as we shall see, $|\xi_p| \le \text{Const} \cdot p^{-(\alpha_f - 1)}$ as p tends to infinity. In terms of ξ_p we have

$$\begin{aligned}\left(1 + \frac{\xi_p}{p^3}\right)^{-p} = {} & \frac{1}{p}\sum_{p_1=1}^{p-1} f(\gamma) + \frac{1}{p}\sum_{p_1=1}^{p-2} \tilde{f}(\gamma)(\Lambda_{p_1}(a_{p-1})-1) \\ & + \frac{1}{p}\sum_{p_1+p_2=p} f(\gamma)(\Lambda_{p_1}(a_{p-1})-1)(\Lambda_{p_2}(a_{p-1})-1).\end{aligned}$$

Or in a better form,

$$\begin{aligned} & \underbrace{p\left(1 - \frac{1}{p}\sum_{p_1=1}^{p-1} f(\gamma)\right)}_{R_p^{(1)}} + \underbrace{p\left(\left(1 + \frac{\xi_p}{p^3}\right)^{-p} - 1\right)}_{R_p^{(2)}} - \underbrace{\sum_{p_1=1}^{p-2} \tilde{f}(\gamma)\left(\prod_{p_1<q<p}\left(1 + \frac{\xi_q}{q^3}\right)^{p_1} - 1\right)}_{Q_p} \\ & = \underbrace{\sum_{p_1+p_2=p} f(\gamma)\left(\prod_{p_1<q<p}\left(1 + \frac{\xi_q}{q^3}\right)^{p_1} - 1\right)\cdot\left(\prod_{p_2<q<p}(1 + \frac{\xi_q}{q^3})^{p_2} - 1\right)}_{N_p}. \end{aligned} \tag{2.1}$$

We shall derive a linear system for ξ_p from the above expression. First observe that for small values of p_1, the summand in Q_p are of order 1 and therefore the main order of Q_p is not linear in ξ_q, $q \le p-1$. However as we shall show in Sect. 3, we have

$$p(Q_p - Q_{p-1}) = \frac{p}{p-1}\xi_{p-1} - \frac{1}{p}\sum_{q=2}^{p-1}\xi_q G(q/p) + O(\log p/p),$$

where $G(\gamma) = \int_0^1 \tilde{f}'(t\gamma)t^2 dt$ and $O(\log p/p)$ denotes terms of higher order of smallness in p. As for $R_p^{(1)}$, since by assumption f is a polynomial, it follows (by direct summation) that

$$p(R_p^{(1)} - R_{p-1}^{(1)}) = O\left(\frac{1}{p}\right).$$

Similarly

$$p(R_p^{(2)} - R_{p-1}^{(2)}) = -\xi_p + \frac{p}{p-1}\xi_{p-1} + O\left(\frac{1}{p^2}\right).$$

The term N_p will be estimated in Sect. 5 and it is of higher order of smallness in p. Put all these considerations together, we see that (2.1) is equivalent to

$$\xi_p = \frac{1}{p}\sum_{q=2}^{p-1} G(q/p)\xi_q + h_p, \tag{2.2}$$

where h_p are of higher order in p.

The main difficulty of estimating ξ_p by using (2.2) is "loss of control of constants". To explain this problem, take our special function $f(\gamma) = 6\gamma^2 - 10\gamma + 4$ and this gives $G(\gamma) = 6\gamma - 4$. It is enough to consider the system

$$\xi_p = \frac{1}{p}\sum_{q=2}^{p-1} G(q/p)\xi_q = \frac{1}{p}\sum_{q=2}^{p-1}(6q/p - 4)\xi_q.$$

Suppose we want to prove by induction that $|\xi_r| \le Ar^{-\alpha}$ for some $\alpha \ge 0$ and $A > 0$. Then at step p we will get

$$\left|\xi_p\right| p^\alpha \le C_p := A\frac{1}{p}\sum_{q=2}^{p-1} |6q/p - 4| \left(\frac{q}{p}\right)^{-\alpha}.$$

As $p \to \infty$, clearly

$$C_p \to A\int_0^1 |6\gamma - 4|\,\gamma^{-\alpha} d\gamma \ge A\int_0^1 |6\gamma - 4|\, d\gamma > A.$$

In other words we will not be able to justify the inductive hypothesis for p sufficiently large. For general function $\tilde{f}$ one can show easily that the integral of G over [0, 1] is -1 (assuming the integral of $\tilde{f}$ is 2). This implies that $\int_0^1 |G(\gamma)|\, d\gamma \ge 1$ and therefore this "loss of control of constants" problem is generic.[1]

[1] Even if $G(\gamma) \le 0$ and $\int_0^1 G = -1$, due to the nonlinear corrections h_p in (2.2), we still have "loss of control of constants".

To solve this problem, we will first prove in Sect. 4 a stability theorem concerning the linear system (2.2). And instead of inducting on ξ_p, we shall induct on h_p. The stability theorem in Sect. 4 gives us bounds on ξ_p by using the (inductively assumed) bounds on h_p. Since h_{p+1} is bounded by quadratic functions of all $\xi_q, q \le p$, the bounds on ξ_q then produce a strong decay estimate on h_{p+1} (see Lemma 5.1). By using a slightly weaker induction hypothesis on h_p (relative to the strong decay estimate), we can justify our inductive bound at step $p+1$ at the sacrifice of assuming p to be sufficiently large. We are able to close our argument because of the genuine nonlinear nature of h_p.

3 The Estimates of $Q_p - Q_{p-1}$

In this section we give the technical estimates of $Q_p - Q_{p-1}$. By definition of Q_p, we have

$$\begin{aligned} Q_p - Q_{p-1} &= \sum_{p_1=1}^{p-2} \tilde{f}(\gamma) \cdot \left(\prod_{p_1<q<p} \left(1+\frac{\xi_q}{q^3}\right)^{p_1} - \prod_{p_1<q<p-1} \left(1+\frac{\xi_q}{q^3}\right)^{p_1} \right) \\ &\quad + \sum_{p_1=1}^{p-3} (\tilde{f}(\gamma) - \tilde{f}(\gamma')) \cdot \left(\prod_{p_1<q<p-1} \left(1+\frac{\xi_q}{q^3}\right)^{p_1} - 1 \right) \\ &= \text{(I)} + \text{(II)}. \end{aligned}$$

We shall show that in the main order of magnitude, we have

$$\text{(I)} \approx \frac{\xi_{p-1}}{p-1},$$

and

$$\text{(II)} \approx -\frac{1}{p^2} \sum_{q=2}^{p-1} \xi_q G(q/p).$$

Throughout this section we make the following ansatz on $\xi_q, q < p$:

Assumption 3.1 Let A_1, A_2, A_3 be positive constants. Let $N_0 \ge \max\{3, A_1\}$ be a positive integer, and $p > N_0$ such that:

(1) $|\xi_q| \le A_1$, for any $N_0 \le q < p$.
(2) $|\xi_q| \le A_3$, for any $1 \le q \le N_0$.
(3) $|\prod_{p_1<q\le N_0} (1+\frac{\xi_q}{q^3})^{p_1}| \le A_2$, for any $1 \le p_1 < N_0$.

Denote $G(\gamma) = \int_0^1 \tilde{f}'(\gamma t) t^2 dt$, we have the following main lemma.

Lemma 3.2 (Main lemma) *Let Assumption 3.1 hold. Then there exists a constant* $C = C(\|\tilde{f}\|_\infty, \|\tilde{f}'\|_\infty, \|\tilde{f}''\|_\infty, A_1, A_2, A_3, N_0)$, *such that*

$$\left| p(Q_p - Q_{p-1}) - \frac{p}{p-1}\xi_{p-1} + \frac{1}{p}\sum_{q=2}^{p-1} G(q/p)\xi_q \right| \le C \frac{\log p}{p}.$$

Proof Recall that $Q_p - Q_{p-1} = \text{(I)} + \text{(II)}$ and we will estimate (I) and (II) separately. First note that $\int_0^1 \tilde{f}(\gamma)\gamma d\gamma = 1$. Using this fact we have

$$\begin{aligned}
\text{(I)} &= \sum_{p_1=1}^{p-2} \tilde{f}(\gamma)\left(\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1} - 1\right)\cdot\left(1-\left(1+\frac{\xi_{p-1}}{(p-1)^3}\right)^{-p_1}\right) \\
&\quad + \sum_{p_1=1}^{p-2} \tilde{f}(\gamma)\left(1-\left(1+\frac{\xi_{p-1}}{(p-1)^3}\right)^{-p_1} - \frac{p_1\xi_{p-1}}{(p-1)^3}\right) \\
&\quad + \left(\sum_{p_1=1}^{p-2} \tilde{f}(\gamma)\frac{p_1}{(p-1)^2} - \int_0^1 \tilde{f}(\gamma)\gamma d\gamma\right)\frac{\xi_{p-1}}{p-1} + \frac{\xi_{p-1}}{p-1} \\
&= e_p^{(1)} + e_p^{(2)} + e_p^{(3)} + \frac{\xi_{p-1}}{p-1}.
\end{aligned}$$

`Estimate of` $e_p^{(1)}$: clearly

$$\begin{aligned}
\left|e_p^{(1)}\right| &\le \left\|\tilde{f}\right\|_\infty \sum_{p_1=1}^{p-2}\left|\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1} - 1\right|\cdot\frac{p_1A_1}{(p-1)^3} \\
&\le \left\|\tilde{f}\right\|_\infty \sum_{N_0\le p_1\le p-2}\frac{A_1}{p_1}\cdot\frac{p_1A_1}{(p-1)^3} + \\
&\qquad + \left\|\tilde{f}\right\|_\infty \sum_{1\le p_1<N_0}\left(1+A_2\left(1+\frac{A}{N_0}\right)\right)\cdot\frac{p_1A_1}{(p-1)^3} \\
&\le \left\|\tilde{f}\right\|_\infty\left(\frac{A_1^2}{(p-1)^2} + \frac{(1+A_2(1+\frac{A_1}{N_0}))A_1N_0^2}{2(p-1)^3}\right).
\end{aligned}$$

`Estimate of` $e_p^{(2)}$: first it is rather easy to show that

$$\left|1-\left(1+\frac{\xi_{p-1}}{(p-1)^3}\right)^{-p_1}\right| \le \frac{p_1A_1}{(p-1)^3}.$$

$$\left|\left(1+\frac{\xi_{p-1}}{(p-1)^3}\right)^{p_1} - 1 - \frac{p_1\xi_{p-1}}{(p-1)^3}\right| \le \frac{3p_1^2A_1^2}{2(p-1)^6},$$

and also

$$\left|\left(1+\frac{\xi_{p-1}}{(p-1)^3}\right)^{-p_1} - 1 + \frac{p_1\xi_{p-1}}{(p-1)^3}\right| \le \frac{3p_1^2A_1^2}{2(p-1)^6}.$$

Now it follows easily that

$$\left|e_p^{(2)}\right| \le \left\|\tilde{f}\right\|_\infty \sum_{p_1=1}^{p-2}\frac{3p_1^2A_1^2}{2(p-1)^6}$$

$$\leq \left\| \tilde{f} \right\|_\infty \frac{A_1^2}{2(p-1)^3}.$$

`Estimate of` $e_p^{(3)}$: This estimate actually does not require Assumption 3.1. The only assumption needed here is that $p \geq 4$. Although this is a standard estimate, for the purpose of explicit control of constants, we give the full details here. Clearly

$$\begin{aligned}
&\left| \sum_{p_1=1}^{p-2} \tilde{f}(\gamma) \cdot \frac{p_1}{p-1} \cdot \frac{1}{p-1} - \int_0^1 \tilde{f}(\gamma)\gamma d\gamma \right| \\
&\quad \leq \sum_{p_1=1}^{p-2} \int_{\frac{p_1-1}{p-1}}^{\frac{p_1}{p-1}} \left| \tilde{f}\left(\frac{p_1}{p}\right) \cdot \frac{p_1}{p-1} - \tilde{f}(\gamma)\gamma \right| d\gamma + \int_{\frac{p-2}{p-1}}^{1} \left| \tilde{f}(\gamma) \right| \gamma d\gamma \\
&\quad \leq \sum_{p_1=1}^{p-2} \int_{\frac{p_1-1}{p-1}}^{\frac{p_1}{p-1}} \left\| \tilde{f}' \right\|_\infty \left| \gamma - \frac{p_1}{p} \right| \frac{p_1}{p-1} d\gamma + \left\| \tilde{f} \right\|_\infty \cdot \frac{p-2}{2(p-1)^2} + \frac{1}{p-1} \left\| \tilde{f} \right\|_\infty \\
&\quad \leq \frac{1}{4(p-1)} \left\| \tilde{f}' \right\|_\infty + \frac{3}{2(p-1)} \left\| \tilde{f} \right\|_\infty.
\end{aligned}$$

The estimates of (II) are similar. Write

$$\begin{aligned}
\text{(II)} &= \sum_{p_1=1}^{p-3} \left(\tilde{f}(\gamma) - \tilde{f}(\gamma') + \tilde{f}'(\gamma) \cdot \frac{\gamma}{p-1} \right) \left(\prod_{p_1<q<p-1} \left(1 + \frac{\xi_q}{q^3}\right)^{p_1} - 1 \right) \\
&\quad + \sum_{p_1=1}^{p-3} \left(-\tilde{f}'(\gamma) \cdot \frac{\gamma}{p-1} \right) \cdot \left(\left(1 + \frac{\xi_q}{q^3}\right)^{p_1} - 1 - p_1 \sum_{p_1<q<p} \frac{\xi_q}{q^3} \right) \\
&\quad + \left(\sum_{p_1=1}^{p-3} \left(-\tilde{f}'(\gamma) \cdot \frac{\gamma}{p-1} \right) \cdot \left(p_1 \sum_{p_1<q<p} \frac{\xi_q}{q^3} \right) + \frac{1}{p^2} \sum_{q=2}^{p-2} \xi_q G(q/p) \right) \\
&\quad - \frac{1}{p^2} \sum_{q=2}^{p-2} \xi_q G(q/p) \\
&= e_p^{(4)} + e_p^{(5)} + e_p^{(6)} - \frac{1}{p^2} \sum_{q=2}^{p-2} \xi_q G(q/p).
\end{aligned}$$

`Estimate of` $e_p^{(4)}$ `and` $e_p^{(5)}$: by Taylor expansion on $\tilde{f}$ we have

$$\left| \tilde{f}(\gamma) - \tilde{f}(\gamma') + \tilde{f}'(\gamma) \cdot \frac{\gamma}{p-1} \right| \leq \frac{1}{2} \left\| \tilde{f}'' \right\|_\infty \cdot \frac{\gamma^2}{(p-1)^2}.$$

Then by Lemma 6.5 in the Appendix,

$$\left| e_p^{(4)} \right| \leq \left\| \tilde{f}'' \right\|_\infty \left(\frac{A_1}{4p^2} + \frac{N_0^3}{6(p-1)^2 p^2} \left(1 + A_2 \left(1 + \frac{A_1}{N_0} \right) \right) \right).$$

Similarly the estimate of $e_p^{(5)}$ follows from Lemma 6.7 in the Appendix:

$$\left|e_p^{(5)}\right| \le \left\|\tilde{f}'\right\|_\infty \left(\frac{A^2\log(p-3)}{4p(p-1)} + \frac{1}{p(p-1)}\left(\frac{1+A_2}{2}N_0^2 + \frac{(3A_2+1)A_1+3A_3}{6}N_0\right)\right).$$

`Estimate of` $e_p^{(6)}$: First note that

$$\begin{aligned}
&\left|\frac{1}{p-1}\sum_{1\le p_1<q}\tilde{f}'(\gamma)\gamma^2 - \int_0^{\frac{q}{p}}\tilde{f}'(\gamma)\gamma^2 d\gamma\right| \\
&\quad\le \sum_{p_1=1}^{q-1}\int_{\frac{p_1-1}{p}}^{\frac{p_1}{p}}\left|\frac{p}{p-1}\tilde{f}'\left(\frac{p_1}{p}\right)\cdot\left(\frac{p_1}{p}\right)^2 - \tilde{f}'(\gamma)\gamma^2\right|d\gamma + \int_{\frac{q-1}{p}}^{\frac{q}{p}}\left|\tilde{f}(\gamma)\right|\gamma^2 d\gamma \\
&\quad\le \sum_{p_1=1}^{q-1}\left(\left\|\tilde{f}'\right\|_\infty\frac{p_1^2}{p^3(p-1)} + \left(\left\|\tilde{f}''\right\|_\infty + 2\left\|\tilde{f}'\right\|_\infty\right)\frac{p_1}{p^3}\right) + \left\|\tilde{f}'\right\|_\infty\frac{q^2}{p^3} \\
&\quad\le \left\|\tilde{f}'\right\|_\infty\left(\frac{q^3}{3p^4} + \frac{2q^2}{p^3}\right) + \left\|\tilde{f}''\right\|_\infty\frac{q^2}{2p^3}.
\end{aligned}$$

Then

$$\sum_{p_1=1}^{p-3}\tilde{f}'(\gamma)\cdot\frac{\gamma}{p-1}\sum_{p_1<q<p-1}\frac{p_1\xi_q}{q^3} = \frac{1}{p^2}\sum_{q=2}^{p-2}\xi_q\sum_{1\le p_1<q}\frac{p^3}{q^3}\frac{1}{p-1}\tilde{f}'(\gamma)\cdot\gamma^2$$

Now it follows easily that

$$\begin{aligned}
\left|e_p^{(6)}\right| &= \left|\frac{1}{p^2}\sum_{q=2}^{p-2}\xi_q\sum_{1\le p_1<q}\frac{p^3}{q^3}\frac{1}{p-1}\tilde{f}'(\gamma)\cdot\gamma^2 - \frac{1}{p^2}\sum_{q=2}^{p-2}\xi_q G(q/p)\right| \\
&\le \frac{1}{p^2}\sum_{q=2}^{p-2}|\xi_q|\left(\left\|\tilde{f}'\right\|_\infty\left(\frac{1}{3p}+\frac{2}{q}\right) + \left\|\tilde{f}''\right\|_\infty\frac{1}{2q}\right).
\end{aligned}$$

The lemma is proved. □

4 Behavior of Solutions to the Linear System

In this section we shall study the linear system defined by

$$\xi_p = \frac{1}{p}\sum_{q=2}^{p-1}G(q/p)\xi_q + h_p.$$

As was already mentioned in Sect. 2, this linear system is obtained from the nonlinear relation (2.1). We shall now free ourselves from the constraint that h_p is a quadratic function of all ξ_q, $q \le p$. Instead we assume here that h_p is known. Under some conditions on the function G and the sequence h_p, we show that $\xi_p \approx \text{const}\cdot p^\sigma$, where $\sigma = \sigma(G)$ is to be defined later in this section. We begin with a simple lemma.

Lemma 4.1 *Let $N \geq 1$. Assume $\{C_i\}_{1\leq i\leq N}$, $\{\alpha_i\}_{1\leq i\leq N}$, $\{\sigma_i\}_{1\leq i\leq N}$ are complex numbers such that*:

(1) $\alpha_i \neq \alpha_j$, $\sigma_i \neq \sigma_j$, *if* $i \neq j$.
(2) $\sum_{i=1}^{N} \frac{C_i}{\sigma_k+\alpha_i+1} = 1, \quad \forall 1 \leq k \leq N$.
(3) $\mathrm{Re}(\sigma_k) \leq 0, \forall 1 \leq k \leq N$.

Suppose $M_p \in \mathbb{R}^{N\times N}$, $p \geq 2$ is a sequence of matrices defined by

$$M_p^{(k,j)} = \left(1-\frac{1}{p}\right)^{\alpha_k+1} \delta_{kj} + C_j \cdot \left(1-\frac{1}{p}\right)^{\alpha_j+1} \cdot \frac{1}{p}.$$

Then there exists an a constant $K > 0$ which depends only on α_i, σ_i, C_i, such that

$$\left\| M_p M_{p-1} \cdots M_{q_0+1} M_{q_0} \right\|_2 \leq K,$$

for any $q_0 \geq 2$, $p \geq 3$, where $\|\cdot\|_2$ denotes the matrix spectral norm.

Proof We have

$$\begin{aligned} M_p^{(k,j)} &= \left(1-\frac{\alpha_k+1}{p}\right)\delta_{kj} + \frac{C_j}{p} + O\left(\frac{1}{p^2}\right) \\ &= \delta_{kj} + \frac{C_j-(\alpha_k+1)\delta_{kj}}{p} + O\left(\frac{1}{p^2}\right). \end{aligned}$$

Consider the matrix $\tilde{M}$ defined by

$$\tilde{M}^{(k,j)} = C_j - (\alpha_k+1)\delta_{kj}.$$

It is not difficult to show that the equation for eigenvalues of $\tilde{M}$ $\det(\tilde{M}-\lambda I) = 0$ is equivalent to:

$$\sum_{i=1}^{N} \frac{C_i}{\lambda+\alpha_i+1} = 1.$$

By our assumption $\tilde{M}$ has N distinct eigenvalues $\sigma_1, \ldots, \sigma_N$. Let D be the diagonal matrix such that $D_{ii} = \sigma_i$. It follows easily that there exist $S, S^{-1} \in \mathbb{C}^{N\times N}$, such that

$$M_p = S\left(I + \frac{1}{p}D + O\left(\frac{1}{p^2}\right)\right)S^{-1}.$$

Since $\max_{1\leq k\leq N} \mathrm{Re}(\sigma_k) \leq 0$, we have

$$\left\| I + \frac{1}{p}D + O\left(\frac{1}{p^2}\right)\right\|_2 \leq 1 + \frac{C_1}{p^2},$$

where C_1 is a constant independent of p. Now the theorem follows by using the submultiplicative property of the matrix spectral norm. □

In what follows, we shall assume that $G(\cdot)$ is a finite sum of (generalized) monomials, i.e.

$$G(\gamma)=\sum_{i=1}^{N} C_i \gamma^{\alpha_i}, \tag{4.1}$$

where α_i, C_i are complex numbers, in particular, note that α_i need not be integers. Here for $0<\gamma\le 1$ and $\alpha=a+bi$, γ^{α} is simply defined as $\exp\{(a+bi)\log\gamma\}$ where $\log\gamma$ is real-valued. The main assumption on G is the following:
Assume that the equation in λ

$$\sum_{i=1}^{N} \frac{C_i}{\lambda+\alpha_i+1}=1, \tag{4.2}$$

has N distinct roots in $\mathbb{C}$.
Denote the N distinct roots by $\sigma_1,\dots,\sigma_N$, and define

$$\sigma(G)=\max_{1\le i\le N} \operatorname{Re}(\sigma_i). \tag{4.3}$$

σ will be called the characteristic exponent of G. We prove the following theorem concerning a linear recurrent system generated by G:

Theorem 4.2 *Assume that G satisfies* (4.1) *and* (4.2). *Consider the linear recurrent system defined by*:

$$\xi_p=\frac{1}{p}\sum_{q=2}^{p-1} G(q/p)\xi_q, \quad p\ge 3,$$

with ξ_2 as a parameter. Then there exists a constant C depending only on G such that

$$\left|\xi_p\right|\le C\cdot p^{\sigma}\cdot|\xi_2|, \quad \forall p>1, \tag{4.4}$$

where the characteristic exponent $\sigma=\sigma(G)$ is defined in (4.3).

Proof We begin with a simple observation. Suppose $\operatorname{Re}(\sigma_{i_0})=\sigma(G)$ and consider

$$\tilde{G}(\gamma)=\sum_{i=1}^{N} C_i \gamma^{\alpha_i-\sigma_{i_0}}.$$

Then $\tilde{G}(\gamma)$ also satisfies the assumption (4.2) with $\tilde{\alpha}_i=\alpha_i-\sigma_{i_0}$. Also $\sigma(\tilde{G})=0$, $\tilde{\xi}_p=p^{-\sigma_{i_0}}\xi_p$, where $\tilde{\xi}_p$ is generated by $\tilde{G}$. With this simple observation, it suffices for us to prove the theorem assuming $\sigma(G)=0$. Assume this is the case and define the moments of ξ_p by:

$$B_p^{(k)}=\sum_{q=2}^{p} \xi_q q^{\alpha_k},$$

where α_k is defined in the definition of G (see (4.1)). Then obviously we have

$$\xi_p=\sum_{k=1}^{N} \frac{C_k}{p^{\alpha_k+1}} B_{p-1}^{(k)}. \tag{4.5}$$

The recurrent formula for $B_p^{(k)}$ follows easily:

$$B_p^{(k)} = B_{p-1}^{(k)} + p^{\alpha_k} \sum_{j=1}^{N} \frac{C_j}{p^{\alpha_j+1}} B_{p-1}^{(j)}.$$

Now if our bound on ξ_p (4.4) is correct, then heuristically $B_p^{(k)}$ grows as $p^{\mathrm{Re}(\alpha_k)+1}$. This motivates us to define the scaled variables

$$\tilde{B}_p^{(k)} = p^{-\alpha_k-1} B_p^{(k)}.$$

For $\tilde{B}_p^{(k)}$ we have the recurrent relation:

$$\tilde{B}_p^{(k)} = \sum_{j=1}^{N} M_p^{(k,j)} \tilde{B}_{p-1}^{(j)},$$

where the matrix $M_p^{(k,j)}$ is given by:

$$M_p^{(k,j)} = \left(1 - \frac{1}{p}\right)^{\alpha_k+1} \delta_{kj} + C_j \cdot \left(1 - \frac{1}{p}\right)^{\alpha_j+1} \cdot \frac{1}{p}.$$

Note that the matrix M_p is the same as in Lemma 4.1. Now denote the vector $\tilde{B}_p = (\tilde{B}_p^{(1)}, \ldots, \tilde{B}_p^{(N)})^T$. Then we have

$$\begin{aligned}\left\|\tilde{B}_p\right\|_2 &= \left\|\left(M_p M_{p-1} \cdots M_3 M_2\right) \tilde{B}_1\right\|_2 \\ &\le \left\|M_p M_{p-1} \cdots M_3 M_2\right\|_2 \left\|\tilde{B}_1\right\|_2.\end{aligned}$$

By Lemma 4.1 we have for some constant C depending only on G such that

$$\left\|M_p M_{p-1} \cdots M_3 M_2\right\|_2 \le C.$$

This immediately implies that $\tilde{B}_p$ is uniformly bounded for all $p > 1$. The desired bound on ξ_p then follows by using (4.5). Our theorem is proved. □

The next lemma can be regarded as an inhomogeneous version of Theorem 4.2. It states that the solution to the linear recurrent system generated by G is stable under sufficiently small perturbations.

Lemma 4.3 *Assume G satisfies* (4.1) *and* (4.2), *with the characteristic exponent $\sigma = \sigma(G)$ defined in* (4.3). *Consider the linear recurrent system defined by*:

$$\xi_p = h_p + \frac{1}{p} \sum_{q=2}^{p-1} G(q/p) \xi_q, \quad p \ge 3,$$

where ξ_2 is a parameter, and assume that for some positive constant C_1 and ϵ, the sequence h_p satisfies

$$\left|h_p\right| \le C_1 \cdot p^{-\epsilon+\sigma}, \quad \forall p \ge 2.$$

Then there exists a constant K independent of p, such that

$$\left|\xi_p\right| \le K C_1 \cdot p^{\sigma} \cdot |\xi_2|, \quad \forall p \ge 2. \tag{4.6}$$

Proof Without loss of generality assume $\sigma(G) = 0$ (see the beginning of the proof of Theorem 4.2). Define $B_p^{(k)}$, $\tilde{B}_p^{(k)}$, $\tilde{B}_p$ and $M_p^{(k,j)}$ as in the proof of Theorem 4.2. Clearly then

$$\xi_p = h_p + \sum_{k=1}^{N} C_k \left(1 - \frac{1}{p}\right)^{\alpha_k+1} \tilde{B}_{p-1}^{(k)}. \tag{4.7}$$

Denote $g_p = p^{-1} h_p (1, \dots, 1)^T$. Then for $\tilde{B}_p$ we have

$$\begin{aligned} \tilde{B}_p &= M_p \tilde{B}_{p-1} + g_p \\ &= M_p (M_{p-1} \tilde{B}_{p-2} + g_{p-1}) + g_p \\ &= \cdots = \sum_{p_1=3}^{p} \left(M_p M_{p-1} \cdots M_{p_1+1}\right) g_{p_1} + \left(M_p M_{p-1} \cdots M_3\right) \tilde{B}_2. \end{aligned}$$

By our assumption on h_{p_1}, we have

$$\left\|g_{p_1}\right\|_2 \le \frac{\sqrt{N} C_1}{p_1^{1+\epsilon}}, \quad \forall 3 \le p_1 \le p.$$

By Lemma 4.1, we have for some constant $\tilde{K} > 0$,

$$\left\|\left(M_p M_{p-1} \cdots M_{p_1+1}\right)\right\|_2 \le \tilde{K}, \quad \forall p_1, p \ge 2.$$

It follows easily that

$$\left\|\tilde{B}_p\right\|_2 \le \tilde{K} \sum_{p_1=3}^{p} \frac{\sqrt{N} C_1}{p_1^{1+\epsilon}} + \tilde{K} C_2 \le K C_1,$$

where C_2 is a constant independent of p. Now use (4.7) to conclude the proof of the lemma. The lemma is proved. □

5 Estimate of the Nonlinear Term and Proof of the Main Theorem

In this section we shall estimate the nonlinear term N_p. We have the following lemma.

Lemma 5.1 *Assume $0 < \sigma < 1$. Let $(\xi_q)_{q \ge 2}$ be a sequence of numbers such that*

$$\left|\prod_{p_1 < q < N_0} \left(1 + \frac{\xi_q}{q^3}\right)^{p_1}\right| \le C_1, \quad \forall 1 \le p_1 < N_0.$$

and

$$\left|\xi_q\right| \le C_2 q^{-\sigma}, \quad \forall N_0 < q < p,$$

where $N_0 \geq \max\{C_2^{\frac{1}{1+\sigma}}, (4C_2)^{\frac{1}{3+\sigma}}\}$. *Let* $p \geq 2N_0$ *and* N_p *be defined by*

$$N_p = \sum_{p_1+p_2=p} f(\gamma)\left(\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1}-1\right)\cdot\left(\prod_{p_2<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_2}-1\right).$$

Then we have

$$|N_p| \leq \left\|\tilde{f}\right\|_\infty\left(\frac{(1+C_1+C_2)2N_0^2C_2}{(p-N_0)^{2+\sigma}}+\frac{4C_2^2C_\sigma}{p^{1+2\sigma}}\right),$$

where

$$C_\sigma = \int_0^1 \gamma^{-\sigma}(1-\gamma)^{-\sigma}d\gamma.$$

Proof First since we have $N_0 \geq \max\{(\frac{2C_2}{2+\sigma})^{\frac{1}{1+\sigma}}, (4C_2)^{\frac{1}{3+\sigma}}\}$, therefore for $r \geq N_0$,

$$\left|\prod_{r<q<p}\left(1+\frac{\xi_q}{q^3}\right)^r-1\right| \leq \frac{2C_2}{2+\sigma}\cdot\frac{1}{r^{1+\sigma}}\cdot\left(1-\left(\frac{r}{p}\right)^{2+\sigma}\right).$$

For $1 \leq r < N_0$, clearly

$$\left|\prod_{r<q<p}\left(1+\frac{\xi_q}{q^3}\right)^r-1\right| \leq 1+C_1+\frac{2C_1C_2}{(2+\sigma)N_0^{1+\sigma}}.$$

For $p-N_0 < p_2 < p$, we get

$$\left|\prod_{p_2<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_2}-1\right| \leq \frac{2N_0C_2}{p_2^{2+\sigma}}.$$

Now we have

$$\begin{aligned}
|N_p| &\leq \left\|\tilde{f}\right\|_\infty\left(\sum_{p_1<N_0}\left|\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1}-1\right|\cdot\left|\prod_{p_2<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_2}-1\right|\right.\\
&\quad\left.+\sum_{N_0\leq p_1\leq\frac{p}{2}}\left|\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1}-1\right|\cdot\left|\prod_{p_2<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_2}-1\right|\right)\\
&\leq \left\|\tilde{f}\right\|_\infty\left(\sum_{p_1<N_0}\left(1+C_1+\frac{2C_1C_2}{(2+\sigma)N_0^{1+\sigma}}\right)\cdot\frac{2N_0C_2}{p_2^{2+\sigma}}\right.\\
&\quad\left.+\sum_{N_0\leq p_1<\frac{p}{2}}\left(\frac{2C_2}{2+\sigma}\right)^2\frac{1}{p_1^{1+\sigma}p_2^{1+\sigma}}\left(1-\left(\frac{p_1}{p}\right)^{2+\sigma}\right)\cdot\left(1-\left(\frac{p_2}{p}\right)^{2+\sigma}\right)\right)\\
&\leq \left\|\tilde{f}\right\|_\infty\left(\frac{(1+C_1+C_2)2N_0^2C_2}{(p-N_0)^{2+\sigma}}+\frac{4C_2^2C_\sigma}{p^{1+2\sigma}}\right).
\end{aligned}$$

□

Proof of the Main Theorem We begin by observing that $\alpha_f = 1 - \sigma(G)$. By Assumption 1.1 we have $0 < -\sigma(G) < 1$. Our inductive assumption is that for $r < p$, we have

$$\xi_r = h_r + \frac{1}{r}\sum_{q=2}^{r-1} G(q/r)\xi_q,$$

and for some positive ϵ, $0 < \epsilon < \frac{1}{2}\min\{1 + \sigma(G), -\sigma(G)\}$,

$$|h_r| \leq C_4 r^{\sigma(G)-\epsilon}.$$

We shall assume that the inductive assumption is justified for some p_0 sufficiently large. Now assume $p \geq p_0$. Lemma 4.3 gives us

$$|\xi_r| \leq K \cdot C_4 r^{\sigma(G)}, \qquad \forall r \leq p-1.$$

At step p, define

$$\hat{\xi}_p = -p^2\left(\left(1 + \frac{\xi_p}{p^3}\right)^{-p} - 1\right).$$

Then we have (see Sect. 2)

$$\hat{\xi}_p = p(R_p^{(1)} - R_{p-1}^{(1)}) - \left(p(Q_p - Q_{p-1}) - \frac{p}{p-1}\hat{\xi}_{p-1}\right) - p(N_p - N_{p-1}).$$

By Lemma 3.2

$$\left|p(Q_p - Q_{p-1}) - \frac{p}{p-1}\hat{\xi}_{p-1} + \frac{1}{p}\sum_{q=2}^{p-1} G(q/p)\xi_q\right| \leq \frac{C_{\epsilon_1}}{p^{1-\epsilon_1}},$$

where C_{ϵ_1} depends only on ϵ, ϵ_1, α, C_4, K and the function f. The parameter $0 < \epsilon_1 < 1$ can be taken arbitrarily small (but C_{ϵ_1} will be large). Take $\epsilon_1 < 1 + \sigma(G) - 2\epsilon$, then by Lemma 5.1 we get

$$\begin{aligned}
&\left|\hat{\xi}_p - \frac{1}{p}\sum_{q=2}^{p-1} G(q/p)\xi_q\right| \\
&\quad \leq \frac{C_{\epsilon_1}}{p^{1-\epsilon_1}} + p\left|N_p(\xi_2, \xi_3, \ldots, \xi_{p-1})\right| + p\left|N_p(\xi_2, \xi_3, \ldots, \xi_{p-1})\right| + \frac{B}{p} \\
&\quad \leq \frac{C_{\epsilon_1}}{p^{1-\epsilon_1}} + D_2 p^{2\sigma(G)} + \frac{B}{p} \leq C_5 p^{\sigma(G)-2\epsilon},
\end{aligned}$$

where D_2, B and C_5 are constants. Now use Lemma 4.3 again to have that

$$\left|\hat{\xi}_p\right| \leq K \cdot C_5 p^{\sigma(G)}.$$

Then by Lemma 6.2, we have

$$\left|h_p\right| \leq \left|\xi_p - \hat{\xi}_p\right| + \left|\hat{\xi}_p - \frac{1}{p}\sum_{q=2}^{p-1} G(q/p)\xi_q\right|$$

$$\leq \frac{C_6}{p^2} + C_5 p^{\sigma(G)-2\epsilon} \leq C_4 p^{\sigma(G)-\epsilon}.$$

where the last inequality follows if we take p_0 to be sufficiently large. We have proved the inductive hypothesis for all p. Now the desired uniform bounds on ξ_p follows again from the stability Lemma 4.3. The main theorem is proved. □

Remark 5.2 *A technical assumption is used in the proof of the main theorem. Namely, we need to justify our inductive assumption up to $p = p_0$. This step is mainly done with the help of numerics. In the special case $f(\gamma) = 6\gamma^2 - 10\gamma + 4$, a good estimate of p_0 is about $p_0 \approx 100$. This is enough for our purposes.*

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Appendix

This appendix is devoted to the elementary estimates needed in previous sections. We note that because of the technical assumption in Sect. 5, we need to get explicit control of constants. For this reason we gather all the needed explicit estimates in this Appendix.

Lemma 6.1 *The following easy elementary inequalities are used*:

- *Suppose $x > -1$, then*

$$\log(1+x) \leq x.$$

- *Suppose $M > N \geq 1$, then for any $\alpha > 0$*

$$\sum_{q=N+1}^{M} \frac{1}{q^{\alpha+1}} \leq \frac{1}{\alpha}\left(\frac{1}{N^\alpha} - \frac{1}{M^\alpha}\right).$$

- *If $0 \leq x \leq 1$, then*

$$e^{\frac{x}{2}} \leq 1 + x.$$

- *If $-\frac{1}{4} \leq x \leq 0$, then*

$$e^{2x} \leq 1 + x.$$

  equivalently we have

$$\log(1+x) \geq 2x.$$

- *If $|x| \leq \frac{1}{2}$, then*

$$e^x \leq 1 + x + x^2.$$

- *If $|x| \leq \frac{1}{4}$, then*

$$\log(1+x) \geq x - x^2.$$

Lemma 6.2 *Let $D_1 > 0$, p_0 is an integer and $p_0 \geq \max\{2\sqrt{D_1}, 7\}$. Suppose $|x| \leq D_1$, Then*

$$\left| p^3 \left(\left(1 + \frac{x}{p^2} \right)^{-\frac{1}{p}} - 1 \right) + x \right| \leq \frac{5x^2}{4p^2} \leq \frac{5D_1^2}{4p^2}, \quad \forall p \geq p_0.$$

Proof One side is trivial:

$$p^3 \left(\left(1 + \frac{x}{p^2} \right)^{-\frac{1}{p}} - 1 \right) + x \geq p^3 \left(\exp\left(-\frac{x}{p^3} \right) - 1 \right) + x \geq 0.$$

For the other side,

$$\begin{aligned} & p^3 \left(\left(1 + \frac{x}{p^2} \right)^{-\frac{1}{p}} - 1 \right) + x \\ & \leq p^3 \left(\exp\left(-\frac{x}{p^3} + \frac{x^2}{p^5} \right) - 1 \right) + x \\ & \leq p^3 \left(-\frac{x}{p^3} + \frac{x^2}{p^5} + \left(\frac{x}{p^3} - \frac{x^2}{p^5} \right)^2 \right) + x \\ & \leq \frac{5x^2}{4p^2}. \end{aligned}$$

□

Lemma 6.3 *Assume that $|\xi_q| \leq A$, for any $q \geq N_0$, where $N_0 \geq \max\{1, A\}$. Then we have*

$$\left| \prod_{p_1 < q < p} \left(1 + \frac{\xi_q}{q^3} \right)^{p_1} - 1 \right| \leq \frac{A}{p_1},$$

for any $p_1 \geq N_0$.

Proof We have by assumption $\frac{|\xi_q|}{q^3} \leq \frac{1}{4}$, therefore

$$\begin{aligned} \prod_{p_1 < q < p} \left(1 + \frac{\xi_q}{q^3} \right)^{p_1} &= \exp\left\{ \sum_{q=p_1+1}^{p-1} p_1 \log\left(1 + \frac{\xi_q}{q^3} \right) \right\} \\ &\leq \exp\left\{ \sum_{q=p_1+1}^{p-1} p_1 \frac{A}{q^3} \right\} \\ &\leq \exp\left\{ \frac{A}{2p_1} \right\} \leq 1 + \frac{A}{p_1}. \end{aligned}$$

Similarly,

$$\begin{aligned}\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1} &= \exp\left\{\sum_{q=p_1+1}^{p-1} p_1 \log\left(1+\frac{\xi_q}{q^3}\right)\right\}\\ &\geq \exp\left\{\sum_{q=p_1+1}^{p-1} p_1 \log\left(1-\frac{A}{q^3}\right)\right\}\\ &\geq \exp\left\{\sum_{q=p_1+1}^{p-1} p_1 \left(-\frac{2A}{q^3}\right)\right\}\\ &\geq \exp\left\{-\frac{A}{p_1}\right\} \geq 1-\frac{A}{p_1}.\end{aligned}$$

□

Lemma 6.4

$$\sum_{N_0\leq p_1\leq p-3}\frac{\gamma^2}{(p-1)^2}\left|\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1}-1\right| \leq \frac{A}{2p^2}.$$

Proof we have

$$\begin{aligned}&\sum_{N_0\leq p_1\leq p-3}\frac{\gamma^2}{(p-1)^2}\left|\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1}-1\right|\\ &\leq \sum_{N_0\leq p_1\leq p-3}\frac{p_1^2}{(p-1)^2p^2}\cdot\frac{A}{p_1}\\ &\leq \frac{A}{2p^2}.\end{aligned}$$

□

Lemma 6.5 *Assume that* $|\xi_q| \leq A_1$*, for any* $q \geq N_0$*, where* $N_0 \geq \max\{1, A_1\}$*. Also assume that for some constant* $A_2 > 0$*,*

$$\prod_{p_1<q\leq N_0}\left(1+\frac{\xi_q}{q^3}\right)^{p_1} \leq A_2, \quad \forall 1\leq p_1\leq N_0.$$

Then we have

$$\sum_{1\leq p_1\leq p-3}\frac{\gamma^2}{(p-1)^2}\left|\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1}-1\right| \leq \frac{A_1}{2p^2}+\frac{N_0^3}{3(p-1)^2p^2}\left(1+A_2\left(1+\frac{A_1}{N_0}\right)\right).$$

Proof We have

$$\sum_{p_1<N_0}\frac{\gamma^2}{(p-1)^2}\left|\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1}-1\right|$$

$$\leq \sum_{p_1<N_0} \frac{p_1^2}{(p-1)^2p^2}\left(A_2 \prod_{N_0<q<p}\left(1+\frac{A_1}{q^3}\right)^{N_0}+1\right)$$

$$\leq \sum_{p_1<N_0} \frac{p_1^2}{(p-1)^2p^2}\left(1+A_2\left(1+\frac{A_1}{N_0}\right)\right)$$

$$\leq \frac{N_0^3}{3(p-1)^2p^2}\left(1+A_2\left(1+\frac{A_1}{N_0}\right)\right).$$

□

Lemma 6.6 *Assume that* $|\xi_q| \leq A$, *for any* $q \geq N_0$, *where* $N_0 \geq \max\{1, A\}$. *Then we have*

$$\left|\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1} - \left(1+p_1\sum_{p_1<q<p}\frac{\xi_q}{q^3}\right)\right| \leq \frac{A^2}{4p_1^2},$$

for any $p_1 \geq N_0$.

Proof First we have

$$\left|\sum_{p_1<q<p}\frac{p_1\xi_q}{q^3}\right| \leq \sum_{p_1<q<p}\frac{p_1 A}{q^3} \leq \frac{A}{2p_1} \leq \frac{1}{2}.$$

Therefore

$$\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1} = \exp\left\{p_1\sum_{p_1<q<p}\log\left(1+\frac{\xi_q}{q^3}\right)\right\}$$

$$\leq \exp\left\{\sum_{p_1<q<p}\frac{p_1\xi_q}{q^3}\right\}$$

$$\leq 1+\sum_{p_1<q<p}\frac{p_1\xi_q}{q^3}+\frac{A^2}{4p_1^2}.$$

Similarly

$$\prod_{p_1<q<p}\left(1+\frac{\xi_q}{q^3}\right)^{p_1} - \left(1+p_1\sum_{p_1<q<p}\frac{\xi_q}{q^3}\right)$$

$$\geq \exp\left\{p_1\sum_{p_1<q<p}\left(\frac{\xi_q}{q^3}-\frac{\xi_q^2}{q^6}\right)\right\} - \left(1+p_1\sum_{p_1<q<p}\frac{\xi_q}{q^3}\right)$$

$$\geq \frac{3}{2}\left(e^{-p_1\sum_{p_1<q<p}\frac{A^2}{q^6}}-1\right)$$

$$\geq \frac{3}{2}\left(e^{-\frac{A^2}{5p_1^4}}-1\right)$$

$$\geq -\frac{3A^2}{10p_1^4} \geq -\frac{A^2}{4p_1^2}.$$

□

Lemma 6.7 *Assume the following*:

(1) $|\xi_q| \leq A_1$, *for any* $q \geq N_0$, *where* $N_0 \geq \max\{1, A_1\}$.
(2) $|\xi_q| \leq A_3$ *for any* $1 \leq q \leq N_0$.
(3) *For some constant* $A_2 > 0$,

$$\prod_{p_1 < q \leq N_0} \left(1 + \frac{\xi_q}{q^3}\right)^{p_1} \leq A_2, \quad \forall 1 \leq p_1 \leq N_0.$$

Then we have

$$\sum_{1 \leq p_1 \leq p-3} \frac{\gamma}{p-1} \left| \prod_{p_1 < q < p} \left(1 + \frac{\xi_q}{q^3}\right)^{p_1} - \left(1 + p_1 \sum_{p_1 < q < p} \frac{\xi_q}{q^3}\right) \right|$$
$$\leq \frac{A_1^2 \log(p-3)}{4p(p-1)} + \frac{1}{p(p-1)} \left(\frac{1 + A_2}{2} N_0^2 + \frac{(3A_2 + 1)A_1 + 3A_3}{6} N_0 \right).$$

Proof We have

$$\sum_{N_0 \leq p_1 \leq p-3} \frac{\gamma}{p-1} \left| \prod_{p_1 < q < p} \left(1 + \frac{\xi_q}{q^3}\right)^{p_1} - \left(1 + p_1 \sum_{p_1 < q < p} \frac{\xi_q}{q^3}\right) \right|$$
$$\leq \sum_{N_0 \leq p_1 \leq p-3} \frac{p_1}{p(p-1)} \cdot \frac{A_1^2}{4p_1^2}$$
$$\leq \frac{A_1^2 \log(p-3)}{4p(p-1)}.$$

Also

$$\left| p_1 \sum_{p_1 < q < p} \frac{\xi_q}{q^3} \right| \leq p_1 \sum_{p_1 < q \leq N_0} \frac{A_3}{q^3} + p_1 \sum_{N_0 < q < p} \frac{A_1}{q^3}$$
$$\leq \frac{A_3}{2p_1} + \frac{p_1 A_1}{2N_0^2}.$$

This gives us

$$\sum_{1 \leq p_1 < N_0} \frac{\gamma}{p-1} \left| \prod_{p_1 < q < p} \left(1 + \frac{\xi_q}{q^3}\right)^{p_1} - \left(1 + p_1 \sum_{p_1 < q < p} \frac{\xi_q}{q^3}\right) \right|$$
$$\leq \frac{1}{p(p-1)} \sum_{1 \leq p_1 < N_0} p_1 \cdot \left(\left| \prod_{p_1 < q < p} \left(1 + \frac{\xi_q}{q^3}\right)^{p_1} - 1 \right| + \left| p_1 \sum_{p_1 < q < p} \frac{\xi_q}{q^3} \right| \right)$$

$$\leq \frac{1}{p(p-1)} \sum_{1 \leq p_1 < N_0} p_1 \cdot \left(1 + A_2\left(1 + \frac{A_1}{N_0}\right) + \frac{A_3}{2p_1} + \frac{p_1 A_1}{2N_0^2}\right)$$

$$\leq \frac{1}{p(p-1)} \left(\frac{1 + A_2}{2} N_0^2 + \frac{(3A_2 + 1)A_1 + 3A_3}{6} N_0\right).$$

□

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